

# Fly High with $\pi$ (3.14159 26535 89793)

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**This document contains project ideas for elementary, middle, and high school students.**

## **Ratios that are good approximations for $\pi$**

The simplest ratio that gives the value of  $\pi$  with 3-digit accuracy is

$$\frac{22}{7} = 3.14$$

A better approximation with 7-digit accuracy is easy to remember, because it contains the first three odd numbers, 1, 3, and 5.

$$\frac{355}{113} = 3.141593$$

What does “digits of accuracy” mean? In digits of accuracy, the leading zeros are not counted. The decimal point has no significance in digits of accuracy. Places of decimal and digits of accuracy are two unrelated ideas. For example:

- 1.234 has 3 places of decimal and 4 digits of accuracy
- 0.00012345 has 8 places of decimal and 5 digits of accuracy

## **Experiments to find $\pi$**

Students can write Lab-reports in groups to learn the format and writing style of lab-reports.

### **Experiment-1: Rolling Disk**

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#### Equipment:

1. Large lid of a bottle, such as peanut butter
2. Masking tape
3. Ruler

Theory:

Circumference of a circle =  $\pi$  times the Diameter of the circle

$$\pi = \text{Circumference divided by the Diameter}$$

Procedure:

1. Put a long strip of masking tape on the floor. Put an ink mark near one end of the tape.
2. Measure the diameter of the lid.
3. Put a narrow strip of masking tape near the periphery of the lid. Put an ink mark on the tape at the rim.
4. Align the ink marks on the lid and the tape on the floor.
5. Roll the lid 10 full turns along the tape on the floor. Put a mark when the two ink marks are aligned after the roll. Repeat 3 times. (The number of turns can be changed to 5 or 20.)
6. Measure the lengths between the starting and ending ink marks
7. Make Table as shown below.

Calculation:

No. of Turns	Diameter	Total Distance	Distance per Turn	Value of $\pi$
10				
10				
10				

1. Divide "Total Distance" by "No. of Turns" to find "Distance per Turn."
2. Divide "Distance per Turn" by "Diameter" to find  $\pi$ .

Discussion:

1. Reliability of the result when the number of turns is changed from 5 to 10 to 20.
2. Multiple observations and averaging.

**Experiment-2: Volume of Cylinder**

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Equipment:

1. An unopened soup can
2. A measuring cylinder partially filled with water
3. Ruler

### Theory:

Volume of a Cylinder =  $\pi$  times Radius of cylinder squared times height of cylinder

$\pi$  = Volume of a cylinder divided by square of the radius divided by height

### Definition of Volume measures:

- A Liter is the volume of a 10cm by 10cm by 10cm cube.
- Volume of a 1m by 1m by 1m cube is 1000 Liter
- A milli-Liter is 1/1000 of Liter
- Volume of a 1cm by 1cm by 1cm cube is 1 milli-Liter
- 1 gallon = 3.785 Liter
- A Liter of water has a mass of 1kg

### Procedure:

1. Measure the radius and height of the soup can. Radius is half of the diameter.
2. Partially fill the measuring cylinder with water and note the graduation mark at the level of water.
3. Fully submerge the soup can in the water and note the graduation mark at the raised level of water.
4. Determine the volume of the displaced water from the difference in the readings from Step-2 and 3.

### Calculation:

Determine the value of  $\pi$  from the equation given in the theory.

### Discussion:

1. What are the sources of error in  $\pi$ ?
2. Maximum possible error in measuring radius = smallest graduation in ruler divided by measured radius (same units).
3. Maximum possible error in measuring height = smallest graduation in ruler divided by measured height (same units)
4. Maximum possible error in measuring volume of displaced water = 2 times the smallest graduation on the measuring cylinder divided by measured volume (why 2 times?)

Total error in the value of  $\pi$  = 2 times error in radius + error in height + error in volume. All pluses give the worst case scenario.

The error in radius is doubled, because radius is squared in the formula for  $\pi$ .

### **Experiment-3: Time-Period of Pendulum**

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### Equipment:

1. Pendulum
2. Stopwatch (Free stopwatch iPad or smartphone Apps are available)
3. Ruler

Theory:

$$T = 2\pi \sqrt{\frac{L}{g}}$$

$T$  = Time period of the pendulum measured in seconds

$L$  = Length of pendulum measured in meters

$g$  = Acceleration of gravity  $9.81\text{m/sec}^2$

$$\pi = \sqrt{\frac{gT^2}{4L}}$$

Procedure:

1. Measure the length of the pendulum.
2. Displace the pendulum through a small angle, say around 10 to 20 degrees and let it swing. One complete oscillation is a swing from one extreme to the other extreme and back to the first extreme.
3. Measure the time taken for 10 complete oscillations by using the stopwatch. Repeat 3 times. (The number of oscillations can be changed to 5 or 20)

Calculation:

No. of Oscillations	Length	Total Time	Time for one oscillation	Value of $\pi$
10				
10				
10				

1. Divide “total time” by “no. of oscillations” to get “Time for one oscillation”.
2. Use formula in the Theory to find  $\pi$ .

Discussion:

1. Reliability of the result for 5, 10, or 20 oscillations.
2. Multiple observations and averaging.

## Formulas for $\pi$

### *Madhava- Leibnitz Series*

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$$\pi = \sqrt{12} \left[ 1 - \frac{1}{3 \times 3} + \frac{1}{5 \times 9} - \frac{1}{7 \times 27} + \frac{1}{9 \times 81} - \frac{1}{11 \times 243} + \frac{1}{13 \times 729} - \dots \right]$$

Note that 3, 9, 27, 81, 243, 729, etc. are all powers of 3.

At this point, it is instructive to introduce the idea of convergence. To understand convergence, let us define a term called “partial sums” and use the symbols

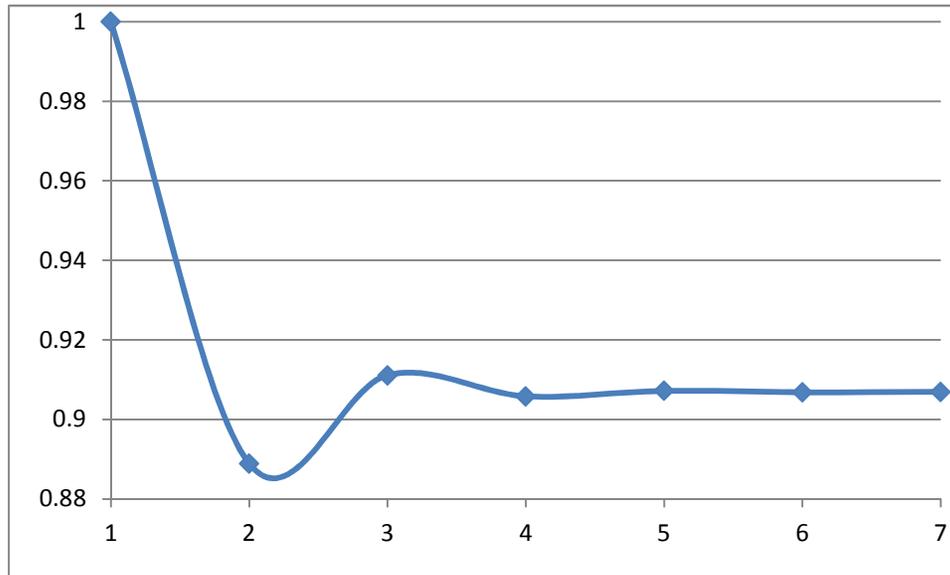
$$S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6 \quad \dots$$

The partial sums are the sums of terms in the series up to 1, 2, 3, 4, etc. number of terms. In the Table below, we have calculated the partial sums up to 7 terms. The numbers show that we already have the sum of the series accurate up to 3 digits as 0.907. From this value, we find  $\pi$  as

$$\pi = \sqrt{12} \times 0.907 = 3.14$$

This shows that Madhava-Leibnitz series gives 3 digits of accuracy in  $\pi$  from the first 7 terms of the series.

No. of terms	Terms	Partial Sums
1	1	1
2	-0.111111111	0.888888889
3	0.022222222	0.911111111
4	-0.005291005	0.905820106
5	0.001371742	0.907191848
6	-0.000374111	0.906817736
7	0.000105519	0.906923255



The plot of the partial sums shows that this series is oscillatory.

### *Euler Series*

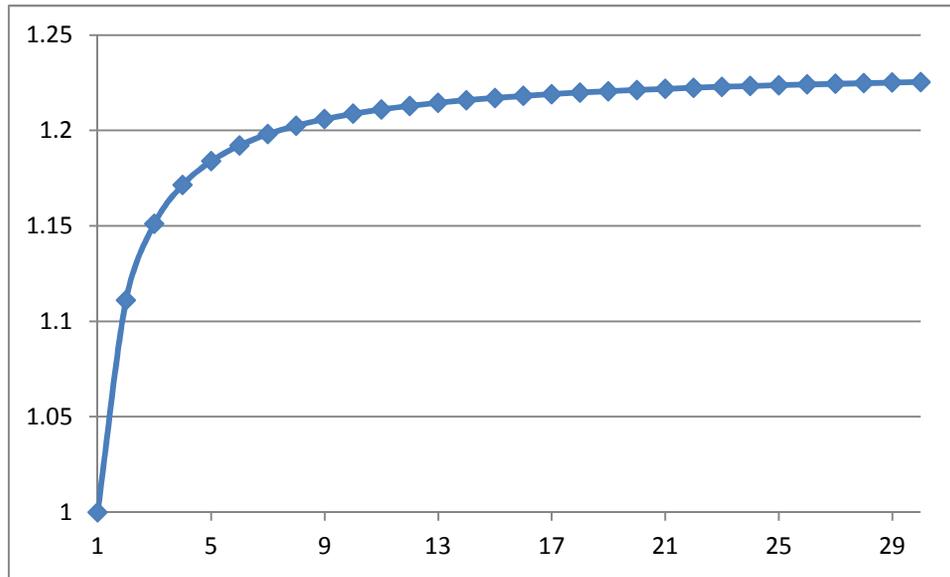
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$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \dots$$

In the Table and the Figure below, we have calculated the partial sums up to 30 terms. The numbers show that we have the sum of the series accurate up to 3 digits as 1.23. From this value, we find  $\pi$  as

$$\pi = \sqrt{8 \times 1.23} = 3.14$$

This shows that Euler series needs 30 terms for 3-digit accuracy in the value of  $\pi$ . Obviously, Euler series has slower convergence compared to Madhava-Leibnitz series.



The plot of the partial sums shows that this series is monotonous.

No. of terms	Terms	Partial Sums
1	1	1
2	0.111111111	1.111111111
3	0.04	1.151111111
4	0.020408163	1.171519274
5	0.012345679	1.183864953
6	0.008264463	1.192129416
7	0.00591716	1.198046576
8	0.004444444	1.20249102
9	0.003460208	1.205951228
10	0.002770083	1.208721311
11	0.002267574	1.210988885
12	0.001890359	1.212879244
13	0.001600000	1.214479244
14	0.001371742	1.215850986
15	0.001189061	1.217040047
16	0.001040583	1.218080629
17	0.000918274	1.218998903
18	0.000816327	1.21981523
19	0.00073046	1.22054569
20	0.000657462	1.221203152
21	0.000594884	1.221798036
22	0.000540833	1.222338869
23	0.000493827	1.222832696
24	0.000452694	1.22328539
25	0.000416493	1.223701883
26	0.000384468	1.22408635
27	0.000355999	1.224442349
28	0.000330579	1.224772927
29	0.000307787	1.225080714
30	0.000287274	1.225367988

## Trigonometric Series

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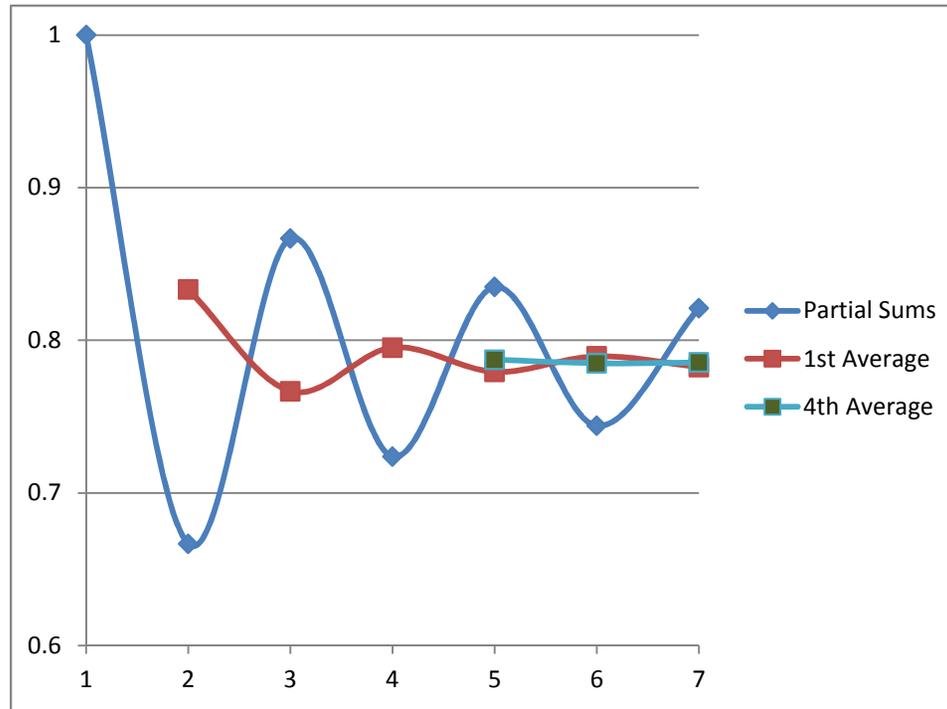
$$\frac{\pi}{4} = \arctan(1.0) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

This is a simple, nice-looking series for calculating  $\pi$ . However, this converges very slowly. To calculate  $\pi$  with 3-digit accuracy, we need close to thousand terms in the series. Hence, we ask the question “Is there any way to accelerate the convergence of a poorly convergent series?” The method described here works particularly well for oscillating series.

In the Table below, we calculated the first 7 partial sums of the series.

No. of terms	Terms	Partial Sums	1st Average of Sums	2nd Average of Sums	3rd Average of Sums	4th Average of Sums
1	1	1				
2	-0.333333333	0.666666667	0.833333333			
3	0.2	0.866666667	0.766666667	0.8		
4	-0.142857143	0.723809524	0.795238095	0.780952381	0.79047619	
5	0.111111111	0.834920635	0.779365079	0.787301587	0.784126984	0.787301587
6	-0.090909091	0.744011544	0.789466089	0.784415584	0.785858586	0.784992785
7	0.076923077	0.820934621	0.782473082	0.785969586	0.785192585	0.785525586

The six 1<sup>st</sup> Averages of Partial Sums are calculated by averaging the two partial sums on the two neighboring rows. The five 2<sup>nd</sup> Averages are calculated by averaging the two 1<sup>st</sup> Averages on the two neighboring rows. The Table shows four such averages. In the diagram below we have plotted the partials sums, 1<sup>st</sup> and 4<sup>th</sup> Averages. The plots show that the repeated averaging process filters out the oscillations in the partial sums.



From the 4<sup>th</sup> average we chose the 3-digit accurate sum of the series as 0.786. From this value, we find  $\pi$  as

$$\pi = 4 \times 0.786 = 3.14$$

This shows that the convergence of a slowly convergent, oscillating series can be accelerated by a simple averaging process and  $\pi$  with 3-digit accuracy can be obtained from the first 7 terms of the series. If you found this interesting, you may like to study more on “Abel Summable” and “Lambert Summable” series, and the notion of “generalized sum” of a series. Techniques for the determination of sums of infinite series are extremely important in Physics, Math, Engineering, and Astrophysics.

### ***John Wallis Product***

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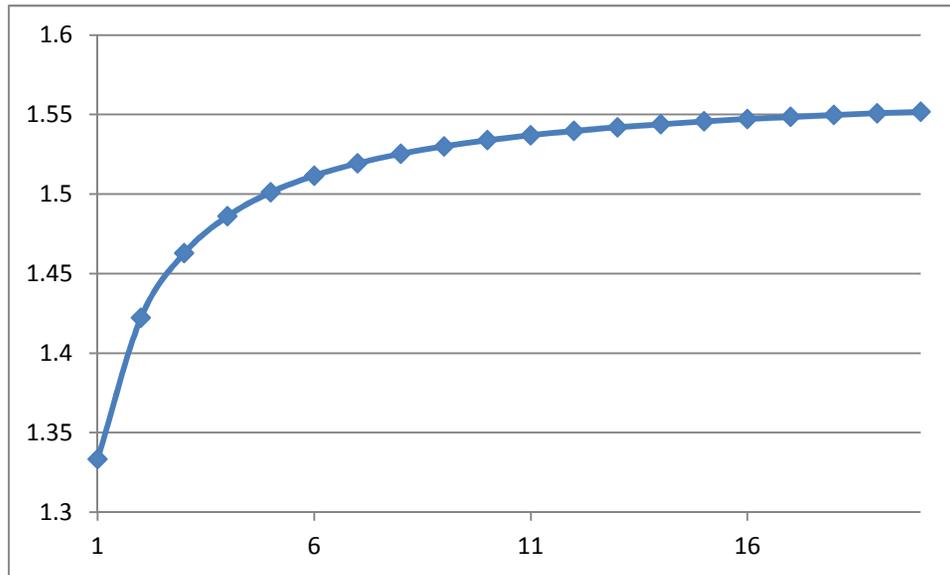
$$\frac{\pi}{2} = \frac{2^2}{1 \times 3} \times \frac{4^2}{3 \times 5} \times \frac{6^2}{5 \times 7} \times \frac{8^2}{7 \times 9} \times \frac{10^2}{9 \times 11} \times \dots$$

Infinite series sometimes come not as a sum, but as a product. For a product series, we calculate partial products instead of partial sums. The Figure and the Table below show the convergence of Wallis’s product series.

No. of terms	Terms	Partial Products
1	1.333333333	1.333333333
2	1.066666667	1.422222222
3	1.028571429	1.462857143
4	1.015873016	1.486077098
5	1.01010101	1.501087977
6	1.006993007	1.511585096
7	1.005128205	1.519336814
8	1.003921569	1.525294998
9	1.003095975	1.530017274
10	1.002506266	1.533851903
11	1.002070393	1.53702758
12	1.00173913	1.539700672
13	1.001481481	1.54198171
14	1.001277139	1.543951035
15	1.001112347	1.545668444
16	1.000977517	1.547179362
17	1.000865801	1.548518911
18	1.000772201	1.549714678
19	1.000693001	1.550788632
20	1.000625391	1.551758481

This is a slowly convergent product as the partial product up to 20 terms gives a value of 1.55. Using this value of partial product, we calculate  $\pi$  as

$$\pi = 2 \times 1.55 = 3.10$$



## Bell-Curve and $\pi$

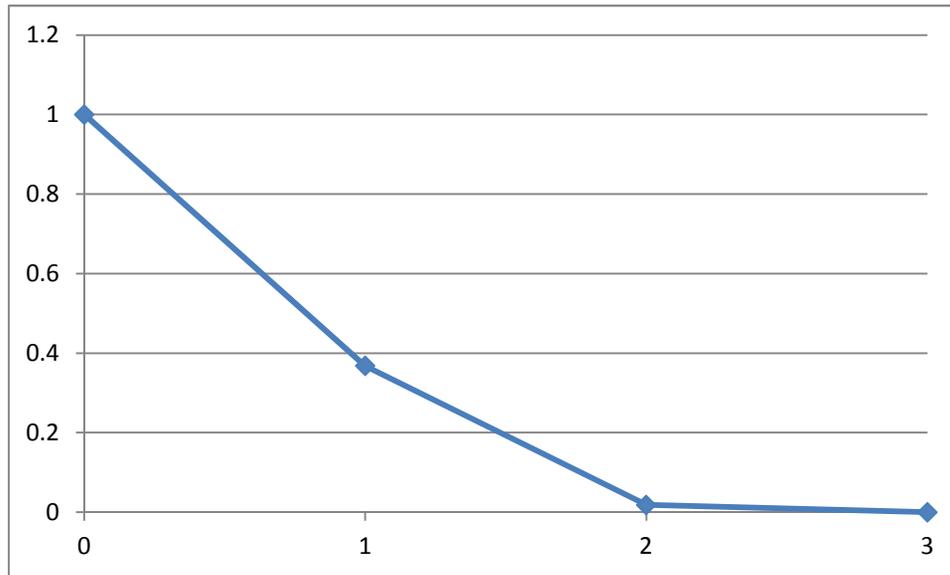
Many random occurrences in nature are often described by the bell-curve. For example, if the average height of a large sample of the population is 5 feet 8 inches, the number of adults with a height of more than 10 feet or with a height of less than 1 foot will be very small. The function underlying the bell-curve is

$$e^{-x^2}$$

One property of this exponential function is

$$\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

The Figure below shows the variation of the exponential function calculated for  $x=0$ ,  $x=1$ ,  $x=2$ , and  $x=3$ .



The integral essentially represents the area under the bell-curve. Instead of extending to infinity, we decided to go only up to  $x=3$ , as the Table shows that the exponential function is pretty close to zero as we reach this value of  $x$ .

To calculate the area under the bell curve we add the area under three trapezoids. The area of a trapezoid can be calculated as

Area of Trapezoid = Average Height multiplied by Width of Trapezoid.

x	Bell Function	Average Height	Width	Trapezoid Area
0	1			
1	0.367879441	0.683939721	1	0.683939721
2	0.018315639	0.19309754	1	0.19309754
3	0.00012341	0.009219524	1	0.009219524
			Total Area =	0.886256785

The average height and the width are shown in the 3<sup>rd</sup> and 4<sup>th</sup> columns of the Table. In the 5<sup>th</sup> column of the Table, the areas of the three trapezoids are shown. The Total Area is calculated as 0.8863. From this value, the value of  $\pi$  is calculated as

$$\sqrt{\pi} = 2 \times 0.8863 = 1.773$$

$$\pi = 3.143$$

To improve the accuracy of the area under the curve, we can make the trapezoids narrow by adding more  $x$ -locations, as show in the Table and the Figure below.

x	Bell Function	Average Height	Width	Trapezoid Area
0	1			
0.5	0.778800783	0.889400392	0.5	0.444700196
1	0.367879441	0.573340112	0.5	0.286670056
1.5	0.105399225	0.236639333	0.5	0.118319666
2	0.018315639	0.061857432	0.5	0.030928716
2.5	0.001930454	0.010123047	0.5	0.005061523
3	0.00012341	0.001026932	0.5	0.000513466
			Total Area=	0.886193623

The total area is calculated as 0.886194. From this value, the value of  $\pi$  is calculated as

$$\sqrt{\pi} = 2 \times 0.886194 = 1.77239$$

$$\pi = 3.14137$$

